

# Conformal flattening maps for the visualization of vessels

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## ABSTRACT

In this note, we employ a conformal mapping technique for flattening branching tubular structures for visualization of MRA and CT volumetric vessel imagery. This may be used for the study of possible vessel pathology (cerebral embolism, cerebral thrombosis, or lung nodules near vessels). Our method is based on a discretization of the Laplace-Beltrami operator to flatten a surface onto a planar polygonal region in an angle-preserving manner.

**Keywords:** flattening maps, shape visualization, conformal mappings, MRA brain imagery

## 1. INTRODUCTION

The technique of surface deformations and flattening can be a useful tool for the visualization of highly undulated and branched surfaces. For example, flattening has been employed in virtual colonoscopy for polyp detection.<sup>5</sup> Flattened representations of the brain surface are also very important in many applications, such as functional magnetic resonance imaging by showing the details of neural activity within the folds of the brain.<sup>1</sup> In this note, we consider the conformal flattening of branched surfaces. Applications include the detection of pathologies from MRA brain images such as cerebral embolism and thrombosis as well as lung nodule detection from CT imagery. In this note, we will illustrate the methodology by flattening Y-shaped tubular surfaces. The technique immediately extends to tubular surfaces with an arbitrary number of branches.

A mapping is conformal if it is a one-to-one mapping between surfaces which preserves angles, and thus preserves the local geometry. Basically, we want to map a section of vessel with branches onto a polygonal region of the plane. In Ref. 5, we have flattened a colon (a topological cylinder) by solving the appropriate Dirichlet problem on the surface defined cutting the tube "lengthwise" from boundary point to another. For the flattening of a Y-shaped vessel, we need to design a more complex cut going through the branch point. With the proper boundary conditions, one can then construct a conformal flattening map by solving a Dirichlet on the cut surface.

We now outline the contents of this note. In Section 2, we give an overview of the flattening method. In Section 3, we describe in detail the numerical method we employ for constructing the flattening map based on a finite element approximation of the Laplace-Beltrami operator on a triangulated surface. Next in Section 4, we apply it to some MRA brain vessel imagery. Finally in Section 5, we discuss some possible extensions and future research directions.

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## 2. APPROACH TO VESSEL FLATTENING

In this section, we outline our approach for the conformal flattening of branched surfaces. Even though it goes through quite generally, we will just give the details for a Y-shaped tubular structure (i.e., with one branch point). The basic theory of Riemann surfaces we are considering here can be found in Ref. 4, and the relevant results on partial differential equations can be found in Ref. 9.

Let  $\Sigma \subset \mathbf{R}^3$  represent an embedded surface (no self-intersections), which is topologically a tube with two tubular branches (see Figure 1) with the three boundary curves (topological circles)  $\sigma_0, \sigma_1$  and  $\sigma_2$ . We want to construct a conformal map,<sup>8</sup>  $f: \Sigma \rightarrow \mathbf{C}$ , which maps  $\Sigma$  to a planar polygonal-shaped region.

The construction of  $f$  begins with the solution of  $u$  to the Dirichlet problem  $\Delta u = 0$  on  $\Sigma \setminus (\sigma_0 \cup \sigma_1 \cup \sigma_2)$ . The boundary conditions are  $u = 0$  on  $\sigma_0$ ,  $u = 1$  on  $\sigma_1$  and  $u = a$  on  $\sigma_2$ , respectively.  $a$  must be chosen carefully to make the branch point  $x_0$  ( $u'(x_0) = 0$ ) be located precisely where the two branch tubes meet.

We then find three smooth curves  $C_0, C_1$  and  $C_2$  on  $\Sigma$  which run from  $x_0$  to  $\sigma_0, \sigma_1$ , and  $\sigma_2$ , respectively (see Figure 1), and such that  $u$  is strictly increasing along the gradient on  $C_1, C_2$  and decreasing along the gradient on  $C_0$ . The curve  $C_i$  meets the boundary  $\sigma_i$  at point  $y_i$  ( $i = 0, 1, 2$ ). Since  $u'(x_0) = 0$ , we can make  $C_1$  and  $C_2$  lie on a line in a neighborhood of  $x_0$ , with  $C_0$  is perpendicular to the line.

These curves define a cut on  $\Sigma$ . Let  $B$  be the oriented boundary of this cut surface, i.e.

$$y_0 \xrightarrow{\sigma_0} y_0 \xrightarrow{-C_0} x_0 \xrightarrow{C_1} y_1 \xrightarrow{\sigma_1} y_1 \xrightarrow{-C_1} x_0 \xrightarrow{C_2} y_2 \xrightarrow{\sigma_2} y_2 \xrightarrow{-C_2} x_0 \xrightarrow{C_0} y_0$$

where “ $-C_i$ ” means running the boundary in the opposite direction of  $C_i$ .

We next compute the function  $v$  which is the harmonic conjugate to  $u$  by solving another Dirichlet problem  $\Delta v = 0$ , given the boundary values of  $v$  satisfies

$$v(\zeta) = \int_{\zeta_0}^{\zeta} \frac{\partial v}{\partial s} ds = \int_{\zeta_0}^{\zeta} \frac{\partial u}{\partial n} ds \quad (1)$$

The proof that the mapping 1-1 is standard.<sup>8</sup>

## 3. NUMERICAL METHOD FOR FLATTENING

In the previous section, we gave the analytical procedure for finding the flattening map  $f$ . Here we will discuss the numerical approximation for the mapping function using a finite element method.<sup>6</sup> In Ref. 5 and 1 we described related methods for colon mapping and brain mapping, respectively. The method used for vessel mapping is similar to them. However, due to the topological differences, the boundary conditions have to be changed.

In this section, we assume that  $\Sigma$  is a triangulated surface and what we are looking for is a flattening function  $f$  that is continuous on  $\Sigma$  and linear on each triangle. It is known<sup>9</sup> that the harmonic function  $u$  is the minimizer of the Dirichlet functional

$$D(u) = \frac{1}{2} \int_{\Sigma} |\nabla u|^2 dS$$

$$u|_{\partial u_0} = 0, u|_{\partial u_1} = 1, u|_{\partial u_2} = \alpha \quad (2)$$

Let  $PL(\Sigma)$  denote the finite dimensional space of piecewise linear functions on  $\Sigma$ . Then we choose a basis  $\{\phi_V\}$  for  $PL(\Sigma)$ . For each vertex  $V \in \Sigma$ , there is a relating basis function such that

$$\begin{aligned} \phi_V(V) &= 1, \\ \phi_V(W) &= 0, W \neq V, \\ \phi_V &\text{ is linear on each triangle.} \end{aligned} \quad (3)$$

Any function  $u \in PL(\Sigma)$  can be written as

$$u = \sum_V u_V \phi_V \quad (4)$$

We can find a minimizer of  $D(u)$  over all  $u \in PL(\Sigma)$  which satisfy the boundary conditions. Matrix  $D_{VW}$  is defined as

$$D_{VW} = \int \int \nabla \phi_V \cdot \nabla \phi_W dS \quad (5)$$

for any pair of vertices  $V$  and  $W$ . It is easily seen that  $D_{VW} \neq 0$  only when  $V$  and  $W$  are connected by an edge in the triangulation. It can be proved<sup>1</sup> that  $u$  is the minimizer of the Dirichlet functional, if for each vertex  $\Sigma \setminus (\sigma_0 \cup \sigma_1 \cup \sigma_2)$ ,

$$\sum_{W \in \Sigma \setminus (\sigma_0 \cup \sigma_1 \cup \sigma_2)} D_{VW} u_W = - \sum_{W \in \sigma_1} D_{VW} - \alpha \sum_{W \in \sigma_2} D_{VW} \quad (6)$$

As shown in Ref. 1, we suppose  $VW$  is an edge belonging to two triangles  $VWX$  and  $VWY$ . A formula from finite-element theory<sup>6</sup> shows that

$$D_{VW} = -\frac{1}{2}(\cot \angle X + \cot \angle Y) \quad (7)$$

where  $\angle X$  is the angle at the vertex  $X$  in the triangle  $VWX$  and  $\angle Y$  is the angle at the vertex  $Y$  in the triangle  $VWY$ . And

$$D_{VV} = - \sum_{W \neq V} D_{VW} \quad (8)$$

The computation for  $v$ , which is conjugate to  $u$ , is similar to that of  $u$ . The boundary values of  $v$  can be obtained from (1) according to the Cauchy-Riemann equations.

#### 4. COMPUTER SIMULATIONS

We tested our algorithm on a data set provided by Surgical Planning Laboratory of Brigham and Women's Hospital. The data set is a  $256 \times 256 \times 60$  MRA brain image.

First, using the segmentation method in Ref. 3 and Ref. 7, we found the surface of a section of vessel. We then generated a triangulation of the surface using the Visualization Toolkit.<sup>10</sup> The triangulated surface was smoothed by using a version of mean curvature flow. The triangles making up the ends of the tubular surface were removed to produce a Y-shaped open-ended tube.

Next, as described in the previous section, we solved the Dirichlet problem for  $u$ , which is the real part of the conformal flattening map. We then found the branch point whose gradient is zero and defined a cut as in Section 2. According to the Cauchy-Riemann equations, we found the boundary values of  $v$  by integral the derivative of  $u$  in the normal direction along the boundary. Similarly, we solved  $v$ , which is the imaginary part of the mapping function. Then, we could conformally flatten the resulting cut surface onto a polygonal region of the plane. We calculated the mean curvature and Gaussian curvature on each point of the surface. We then painted points of the vessel surface according to the curvature and applied the same colors to the flattened image on the plane. This gave a way to visualize the whole structure of the vessel surface at the same time.

In Figure 2 and 3, we showed the triangulated surface of the vessel colored by its mean curvature and Gaussian curvature, respectively. Figures 4 and 5 are the flattened surfaces whose corresponding points have been painted by mean curvature and Gaussian curvature as described before. Finally to illustrate the potential of detecting areas of high curvature, we added a phantom (indicated in green) to the vessel as shown in Figures 6 7 8 and 9.

Although these images show only the curvature characteristics, other geometric quantities, such as the thickness of the vessel wall, can also be visualized by this method.

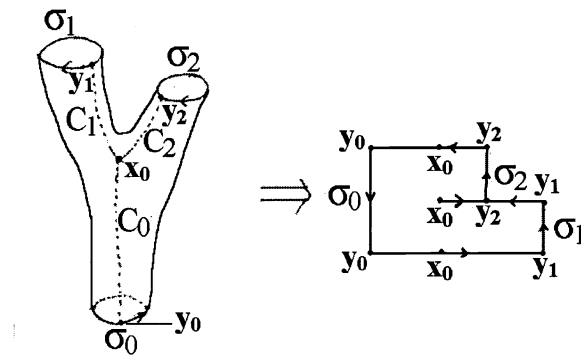


Figure 1: Mapping a Y-shaped vessel onto the plane.

## 5. CONCLUSIONS

In this note, we presented a high-level procedure for the construction of a flattening map of a vessel surface derived from volumetric MRA data. The method is based on conformal geometry and harmonic analysis. We also provided a numerical approximation that finds the mapping based on finite elements.

Additional details on the underlying mathematics, numerics and applications to 3D brain imagery and colonoscopy can be found in Ref. 1 and Ref. 5.

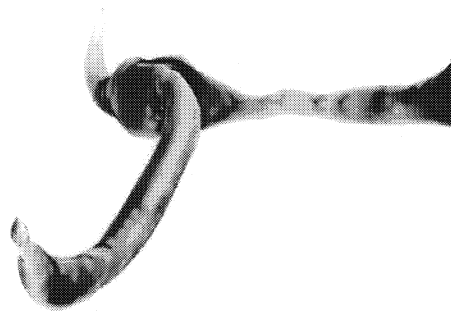
While we have only considered Y-shaped vessels, the method is extendable to the many-branched vessel system of the brain. We intend to treat this in a future paper. Since for surfaces for non-zero Gaussian curvature, angles and area cannot both be preserved, we also intend to consider methodologies based on area-preserving maps of minimal distortion<sup>2</sup> to find the best compromise.

## ACKNOWLEDGMENTS

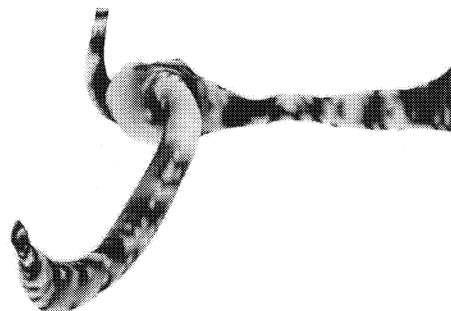
This work was supported in part by grants from the National Science Foundation, NSF-LIS, Air Force Office of Scientific Research, Army Research Office, Coulter Foundation, and MURI Grant.

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**Figure 2:** Mean curvature of the vessel.



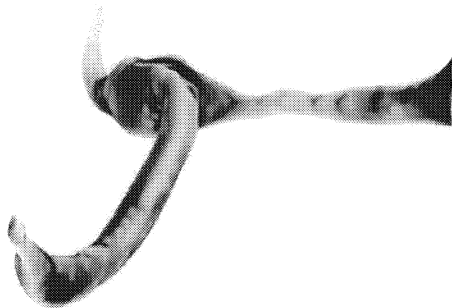
**Figure 3:** Gaussian curvature of the vessel.



**Figure 4:** Points colored according to mean curvature on flattened vessel.



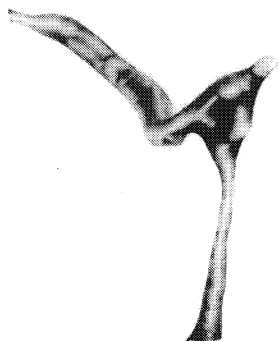
**Figure 5:** Points colored according to Gaussian curvature on flattened vessel.



**Figure 6.** Phantom on vessel. The phantom is indicated green for easier visualization; the other points are colored according to mean curvature.



**Figure 7:** Phantom on vessel, viewing from another angle.



**Figure 8:** Phantom on vessel, viewing from another angle.



**Figure 9:** Flattened version of Figure 6.